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## LETTER TO THE EDITOR

**Extending Bauer's corollary to fractional derivatives**David W Dreisigmeyer<sup>1</sup> and Peter M Young<sup>2</sup><sup>1</sup> Department of Mathematics, Colorado State University, Fort Collins, CO 80523, USA<sup>2</sup> Department of Electrical and Computer Engineering, Colorado State University, Fort Collins, CO 80523, USA

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Online at [stacks.iop.org/JPhysA/37/L117](http://stacks.iop.org/JPhysA/37/L117) (DOI: 10.1088/0305-4470/37/11/L01)**Abstract**

We comment on the method of Dreisigmeyer and Young (2003 *J. Phys. A: Math. Gen.* **36** 8297) to model nonconservative systems with fractional derivatives. It was previously hoped that using fractional derivatives in an action would allow us to derive a single retarded equation of motion using a variational principle. It is proven that, under certain reasonable assumptions, the method of Dreisigmeyer and Young fails.

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**1. Introduction**

In 1931 Bauer proved the following corollary [3]: ‘The equations of motion of a dissipative linear dynamical system with constant coefficients are not given by a variational principle’. There are a few methods that may allow us to get around Bauer's corollary. For example, we could allow additional equations of motion to result. Bateman used this technique in [2]. If we use the Lagrangian

$$L = m\dot{x}\dot{y} + \frac{C}{2}(x\dot{y} - \dot{x}y) - m\omega^2 xy \quad (1)$$

where  $C$  is a constant, we would have the following equations of motion:

$$m\ddot{x} + C\dot{x} + m\omega^2 x = 0 \quad (2)$$

$$m\ddot{y} - C\dot{y} + m\omega^2 y = 0. \quad (3)$$

Bateman's method uses the loophole that Bauer's proof assumed that no additional equations arise.

Riewe pointed out that Bauer's proof also implicitly assumes that all the derivatives are integer ordered [6]. This has led to attempts to use fractional derivatives in the actions to model nonconservative systems [4–6]. Here we will close this second loophole by extending Bauer's corollary to include fractional derivatives.

Our paper is organized as follows. In section 2 we review the background material needed for our result. The extension of Bauer's corollary is proved in section 3. A brief discussion follows in section 4.

## 2. Background material

Here we develop the relevant mathematics for our proof. A fuller discussion of this material can be found in [4]. Fractional derivatives can be defined using the theory of distributions. First, define the generalized functions

$$\Phi_{\alpha}^{+}(t) = \begin{cases} \frac{1}{\Gamma(\alpha)} t^{\alpha-1} & t > 0 \\ 0 & t \leq 0 \end{cases} \quad (4)$$

and

$$\Phi_{\alpha}^{-}(t) = \begin{cases} \frac{1}{\Gamma(\alpha)} |t|^{\alpha-1} & t < 0 \\ 0 & t \geq 0 \end{cases} \quad (5)$$

where  $\Gamma(\alpha)$  is the gamma function. The left fractional derivatives (LFDs) of a function  $q(t)$  is given by

$$\begin{aligned} {}_a \mathbf{D}_t^{\alpha} [q] &:= \Phi_{-\alpha}^{+}(t) * q(t) \\ &= \frac{1}{\Gamma(-\alpha)} \int_a^t q(\tau) (t - \tau)^{-(\alpha+1)} d\tau \end{aligned} \quad (6)$$

where we set  $q(t) \equiv 0$  for  $t < a$ . When  $\alpha = n$ ,  $n$  an integer, (6) becomes

$${}_a \mathbf{D}_t^n [q] = \mathcal{D}^n q \quad (7)$$

where  $\mathcal{D}$  is the generalized derivative. Right fractional derivatives (RFDs) are defined similarly

$$\begin{aligned} {}_t \mathbf{D}_b^{\alpha} [q] &:= \Phi_{-\alpha}^{-}(t) * q(t) \\ &= \frac{1}{\Gamma(-\alpha)} \int_t^b q(\tau) (\tau - t)^{-(\alpha+1)} d\tau \end{aligned} \quad (8)$$

where now  $q(t) \equiv 0$  for  $t > b$ . Instead of (7), we have

$${}_t \mathbf{D}_b^n [q] = (-1)^n \mathcal{D}^n q. \quad (9)$$

In [4] the actions were treated as Volterra series. The Volterra series are a generalization to functionals of the power series of a function. For a functional  $\mathcal{V}[q]$ , define the symmetric kernels

$$K_n^{(s)}(\tau_1, \dots, \tau_n) := \frac{\delta^n \mathcal{V}[q]}{\delta q(\tau_1) \cdots \delta q(\tau_n)}. \quad (10)$$

Now introduce the notation

$$K_n^{(s)} \star q^n := \int_{\tau_1} \cdots \int_{\tau_n} K_n^{(s)}(\tau_1, \dots, \tau_n) q(\tau_n) \cdots q(\tau_1) d\tau_n \cdots d\tau_1. \quad (11)$$

Then  $\mathcal{V}[q]$  can be written as

$$\mathcal{V}[q] = \sum_{n=1}^{\infty} \frac{1}{n!} K_n^{(s)} \star q^n \quad (12)$$

where we set  $K_0^{(s)} \equiv 0$ .

The  $\Phi_\alpha^\pm(t)$  are now treated as kernels in a Volterra series. We can then take the functional derivative of the series to derive our equations of motion. An example should make this clearer. We will restrict our attention to the action

$$\mathcal{V}[q] = \frac{1}{2} K_2 \star q^2 \tag{13}$$

where  $K_2(t, \tau)$  in (13) is an arbitrary kernel, i.e., not necessarily symmetric as in (10). (Equation (13) would be sufficiently general to handle the nonconservative harmonic oscillator.) Now let  $K_2(t, \tau)$  be given by

$$K_2(t, \tau) := m\Phi_{-2}^+(t - \tau) + mC\Phi_{-\gamma}^+(t - \tau) + m\omega^2\Phi_0^+(t - \tau) \tag{14}$$

where  $0 < \gamma < 2$  and  $C$  is a constant. So (13) becomes

$$\mathcal{V}[q] = \frac{m}{2} \int_t \int_\tau [\Phi_{-2}^+(t - \tau) + C\Phi_{-\gamma}^+(t - \tau) + \omega^2\Phi_0^+(t - \tau)] q(\tau) q(t) d\tau dt. \tag{15}$$

The functional derivative of (15) is

$$\begin{aligned} \frac{\delta \mathcal{V}[q]}{\delta q(\rho)} = & \frac{m}{2} \int_\tau \underbrace{[\Phi_{-2}^+(\rho - \tau) + C\Phi_{-\gamma}^+(\rho - \tau) + \omega^2\Phi_0^+(\rho - \tau)]}_{\text{retarded}} q(\tau) d\tau \\ & + \frac{m}{2} \int_t \underbrace{[\Phi_{-2}^+(t - \rho) + C\Phi_{-\gamma}^+(t - \rho) + \omega^2\Phi_0^+(t - \rho)]}_{\text{advanced}} q(t) dt. \end{aligned} \tag{16}$$

If we require the advanced and retarded parts of (16) to vanish separately, we have

$$m[\Phi_{-2}^+(\tau) + C\Phi_{-\gamma}^+(\tau) + \omega^2\Phi_0^+(\tau)] * q(\tau) = 0 \quad (\text{retarded}) \tag{17}$$

$$m[\Phi_{-2}^-(t) + C\Phi_{-\gamma}^-(t) + \omega^2\Phi_0^-(t)] * q(t) = 0 \quad (\text{advanced}) \tag{18}$$

where in (18) we used the fact that  $\Phi_\alpha^+(t - \tau) = \Phi_\alpha^-(\tau - t)$ . This is the method presented in [4] for deriving the equations of motion.

### 3. The result

In section 2 we reviewed the procedure Dreisigmeyer and Young proposed in [4] for deriving a system's equations of motion. From (17) and (18) we see that two equations are actually derived: an advanced one and a retarded one. So this is, effectively, a generalization of Bateman's method (see (1)–(3)). That is, extra equations of motion are allowed to result from the action's variation.

We desire to have a single, retarded equation of motion to result from a variational principle. From (14) we see that the derivative operators are always contained in the  $K_2(\tau_1, \tau_2)$  kernel. Perhaps it is possible to use some kernel other than the  $\Phi_\alpha^\pm(\tau_1 - \tau_2)$  to have a fractional derivative arise from an action's variation? The following theorem shows that this is not possible within the formalism presented in [4].

**Theorem 3.1.** *There does not exist a  $K(\tau_1, \tau_2)$ ,  $\tau_1, \tau_2 \in \mathbb{R}$ , such that the variation of the quantity*

$$\mathcal{V}[q] = \int K(\tau_1, \tau_2) q(\tau_1) q(\tau_2) d\tau_1 d\tau_2 \tag{19}$$

*will result in  ${}_a D_t^\alpha [q]$  for  $\alpha \neq 2n$ ,  $n$  an integer.*

**Proof.** The variation of  $\mathcal{V}[q]$  is given by

$$\frac{\delta\mathcal{V}[q]}{\delta q(\rho)} = [K(\rho, t) + K(t, \rho)] \star q(t). \quad (20)$$

We will assume that

$$[K(\rho, t) + K(t, \rho)] \star q(t) = \Phi_{-\alpha}^+(\rho - t) \star q(t) \quad (21)$$

and arrive at a contradiction. We require that (21) holds for every  $q(t)$ . Then we must have

$$[K(\rho, t) + K(t, \rho)] = \Phi_{-\alpha}^+(\rho - t). \quad (22)$$

Interchanging  $\rho$  and  $t$  in (22) gives us

$$[K(\rho, t) + K(t, \rho)] = \Phi_{-\alpha}^-(\rho - t). \quad (23)$$

Hence, unless  $\Phi_{-\alpha}^\pm(\rho - t)$  is symmetric in  $\rho$  and  $t$ , (22) and (23) cannot both hold. That is, unless  $\alpha = 2n$ ,  $n$  an integer, there does not exist a  $K(\tau_1, \tau_2)$ ,  $\tau_1, \tau_2 \in \mathbb{R}$ , such that (21) holds.  $\square$

Theorem 3.1 shows that, in general, the fractional mechanics formalisms presented in [4–6] cannot derive a single, retarded equation of motion. In order to overcome this difficulty, Riewe suggested approximating RFDs with LFDs [5, 6]. Dreisigmeyer and Young showed in [4] that this is not a sound idea and, instead, allowed for an extra, advanced equation of motion. The latter technique is not, itself, entirely satisfactory.

#### 4. Discussion

Theorem 3.1 shows that some revision of our concept of an action may be in order if we desire a variational principle to work for nonconservative systems. How could we derive a single, retarded equation of motion for systems? Our result holds even if  $K_2(\tau_1, \tau_2)$  is allowed to be complex. We would also require that  $q(\tau_1) = q(\tau_2)$  for  $\tau_1 = \tau_2$  in (19). That is, we do not want to employ Bateman's method, as was done in [4].

One possible method proposed by Tonti [7] (see also [1]) is to use the convolution product in our Lagrangians. This leads to actions of the form

$$\mathcal{V}[q] = \int K(t - \tau_1 - \tau_2)q(\tau_1)q(\tau_2) d\tau_2 d\tau_1 dt. \quad (24)$$

This method does allow the derivation of a single retarded equation of motion for, e.g., the driven nonconservative harmonic oscillator. Unfortunately, it does not seem possible to naturally generalize Tonti's method to higher ordered potentials. That is, using quantities such as

$$\int K(t - \tau_1 - \dots - \tau_n)q(\tau_1) \dots q(\tau_n) d\tau_n \dots d\tau_1 dt \quad (25)$$

in the action will not lead to the correct form for the potential energy terms when  $n > 2$ . This situation should be contrasted with that in [4]. There terms such as

$$\int \Phi_0^+(\tau_1 - \tau_2) \dots \Phi_0^+(\tau_{n-1} - \tau_n)q(\tau_1) \dots q(\tau_n) d\tau_n \dots d\tau_1 \quad (26)$$

were able to treat the potential energy terms correctly. However, as theorem 3.1 demonstrates, the formalism in [4] is unable to deal correctly with the fractional derivative terms.

An interesting feature of (24) versus (19) is the presence of  $t$  in the action along with  $\tau_1$  and  $\tau_2$ . Our stated goal is to derive purely retarded equations of motion using a variational principle. However, to achieve this we need to find the correct kernels for our Volterra series

action. Theorem 3.1 states that kernels of the form  $K_2(\tau_1, \tau_2)$  are not sufficient for our purposes. Equation (24) suggests that we look instead at the kernels  $K(t, \tau_1, \dots, \tau_n)$  for our actions.

It is desirable to be able to use the same type of kernel for the fractional derivatives as well as for the higher ordered potential terms. This assumption is necessary so that the same perturbation of  $q(\tau)$  can be used in the kinetic and potential energy terms of the action. It allows us to reject using kernels of the form  $K(t - \tau_1 - \dots - \tau_n)$ , as in (25). Also, it prevents us from using terms such as (24) for the kinetic energy and terms such as (26) for the potential energy, within the Lagrangian formalism. Finding the correct kernels  $K(t, \tau_1, \dots, \tau_n)$  for our Volterra series actions is a line of research that we are actively pursuing at this time.

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### References

- [1] Arthurs A M and Jones M E 1976 On variational principles for linear initial value problems *J. Math. Anal. Appl.* **54** 840–5
- [2] Bateman H 1931 On dissipative systems and related variational principles *Phys. Rev.* **38** 815–9
- [3] Bauer P S 1931 Dissipative dynamical systems I *Proc. National Academy of Sciences* vol 17 pp 311–4
- [4] Dreisigmeyer D W and Young P M 2003 Nonconservative Lagrangian mechanics: a generalized function approach *J. Phys. A: Math. Gen.* **36** 8297–310
- [5] Riewe F 1996 Nonconservative Lagrangian and Hamiltonian mechanics *Phys. Rev. E* **53** 1890–8
- [6] Riewe F 1997 Mechanics with fractional derivatives *Phys. Rev. E* **55** 3581–92
- [7] Tonti E 1971 On the variational formulation for initial value problems *Ann. Mat. Pura Appl.* **95** 331–59